Space curves: $\vec{r}: \underset{\uparrow \text{ interval}}{I} \longrightarrow \mathbb{R}^n$

def: lim of space curve $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ $t \to t_0$ is component-wise lim if they all exist

$\lim_{t\to a} \vec{r}(t) = \lim_{t\to a} \langle x(t), y(t), z(t)\rangle$

helix: $\vec{r}(t) = \langle \cos(t), \sin(t), t\rangle$ → circle in x-y plane

$\vec{r}(t) = \langle 14 + \sin 120t + \cos t, 4 + \sin 20t + \sin t, \cos 2t$

ex: $\lim_{t\to\infty} \left\langle \frac{1+t^2}{1-t^2}, \arctan(t), \frac{1-e^{-2t}}{t} \right\rangle$

Sol: $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \frac{1+t^2}{1-t^2} = \lim_{t\to\infty} \frac{\frac{1}{t^2}+1}{\frac{1}{t^2}-1} = \frac{0+1}{0-1} = -1$

$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \arctan(t) = \frac{\pi}{2}$

$\lim_{t\to\infty} z(t) = \lim_{t\to\infty} \frac{1-e^{-2t}}{t} \overset{\boxed{H}}{=} \lim_{t\to\infty} \frac{-(-2)e^{-2t}}{1} = \lim_{t\to\infty} 2e^{-2t} = 0$

resultant vector is $\langle -1, \frac{\pi}{2}, 0\rangle$

Continuity: a space curve $\vec{r}(t)$ is cont. at $t=a$ when $\lim_{t\to a} \vec{r}(t) = \vec{r}(a)$

ex: where is $\vec{r}(t) = \left\langle \frac{1+t^2}{1-t^2}, \arctan(t), \frac{1-e^{-2t}}{t} \right\rangle$ continuous?

$\vec{r}(t)$ is continuous at a iff each component is

domains: $x(t): t \neq \pm 1 \Rightarrow (-\infty,-1) \cup (-1,1) \cup (1,\infty)$, $y(t): (-\infty,\infty)$

$z(t): (-\infty,0) \cup (0,\infty)$

$\vec{r}(t)$ is cts on $(-\infty,-1) \cup (-1,0) \cup (0,1) \cup (1,\infty)$

derivative of space curve $\vec{r}(t)$ at $t=a$ is $\boxed{\vec{r}\,'(a) = \lim_{h\to 0} \frac{\vec{r}(a+h) - \vec{r}(a)}{h}}$ ← extremely important

ex: $\vec{r}\,'(t)$ for $\vec{r}(t) = \langle t, t^2, \sqrt{t}\rangle$:

$\vec{r}\,'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} = \lim_{h\to 0} \frac{1}{h}\left(\langle t+h, (t+h)^2, \sqrt{t+h}\rangle - \langle t, t^2, \sqrt{t}\rangle\right)$

$= \lim_{h\to 0} \frac{1}{h} \langle h, 2ht + h^2, \sqrt{t+h} - \sqrt{t}\rangle = \lim_{h\to 0} \left\langle 1, 2t+h, \frac{\sqrt{t+h}-\sqrt{t}}{h} \right\rangle$

$= \left\langle \lim_{h\to 0} 1, \lim_{h\to 0} (2t+h), \lim_{h\to 0} \frac{\sqrt{t+h}-\sqrt{t}}{h} \right\rangle = \langle 1, 2t, \_\rangle$ $\lim_{h\to 0} \frac{\sqrt{t+h}-\sqrt{t}}{h} \cdot \frac{\sqrt{t+h}+\sqrt{t}}{\sqrt{t+h}+\sqrt{t}}$

$= \lim_{h\to 0} \frac{(t+h)-t}{h(\sqrt{t+h}+\sqrt{t})} = \lim_{h\to 0} \frac{h}{h(\sqrt{t+h}+\sqrt{t})} = \frac{1}{\sqrt{t+0}+\sqrt{t}} = \frac{1}{2\sqrt{t}}$ what really happened?

$\lim_{h\to 0} \frac{\vec{r}(t+h)-\vec{r}(t)}{h} = \left\langle \lim_{h\to 0} \frac{x(t+h)-x(t)}{h}, \lim_{h\to 0} \frac{y(t+h)-y(t)}{h} \right\rangle = \langle x'(t), y'(t) \rangle$

derivatives are computed componentwise! ☺ expresses instantaneous velocity in each direction

Properties: $\vec{r}(t)$ and $\vec{s}(t)$ are space curves, $c(t)$ is a scalar fnctn, derivatives exist

① $\frac{d}{dt}[\vec{r}(t)+\vec{s}(t)] = \vec{r}\,'(t) + \vec{s}\,'(t)$

② $\frac{d}{dt}[c(t)\vec{r}(t)] = c'(t)\vec{r}(t) + c(t)\vec{r}\,'(t)$

③ $\frac{d}{dt}[\vec{r}\,'(t)\cdot\vec{s}\,'(t)] = \frac{d}{dt}[\langle x(t), y(t)\rangle \cdot \langle a(t), b(t)\rangle] = \frac{d}{dt}[x(t)a(t)] + \frac{d}{dt}[y(t)b(t)]$

$= (x'(t)a(t) + y'(t)b(t)) + (x(t)a'(t) + y(t)b'(t)) = \langle x', y'\rangle \cdot \langle a, b\rangle + \langle x, y\rangle \cdot \langle a', b'\rangle$

$\Rightarrow \vec{r}\,'(t)\cdot\vec{s}(t) + \vec{r}(t)\cdot\vec{s}\,'$

④ $\frac{d}{dt}[\vec{r}(t)\times\vec{s}(t)] = \vec{r}\,'(t)\times\vec{s}(t) + \vec{r}(t)\times\vec{s}\,'(t)$ ← do not switch around

⑤ $\frac{d}{dt}[\vec{r}(c(t))] = \vec{r}\,'(c(t))\cdot c'(t)$ ← chain rule

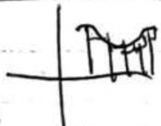
exercise: verify for $\mathbb{R}^3$ curves

$\vec{r}\,'(t)$ = tangent vector to $\vec{r}(t)$ at time t

unit tangent vector is $\frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$ if $\vec{r}\,'(t) \neq \vec{0}$ speed of $\vec{r}(t)$ is $|\vec{r}\,'(t)|$

exercise: prove that if $\vec{r}(t)$ has constant speed, then $\vec{r}(t)$ and $\vec{r}\,'(t)$ are orthogonal

integrals of space curves:

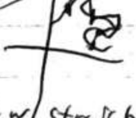


$\int f(t)\,dt = \lim_{n\to\infty} \sum_{k=0}^{n} f(t_k^*)\Delta t_k$

def $\int_{t=a}^{b} \vec{r}(t)\,dt = \left\langle \int_a^b x(t)\,dt, \int_a^b y(t)\,dt, \int_a^b z(t)\,dt \right\rangle$ for $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$

interpretation: $\int_{t=a}^{b} \vec{r}(t)\,dt$ represents displacement

arc length! pick lots of points and add lengths



① approximate w/ straight lines

② sum of lines' lengths is known

these approximations limit to tangent line

arc length $[a,b]$ of $\vec{r}(t) = \int_{t=a}^{b} |\vec{r}\,'(t)|$